Figure 2 shows a computer-calculated plot of log J* against the Stokes number St for poorly conducting particles in the case of potential flow past a sphere of radius R. Curves 1-3 correspond to Re = 10, 10^2, and 10^3. When St = ∞ we have

$$J^* = 2\int_0^1 (\sqrt{1-b^{*2}})^{7/10} b^* db^* = \frac{20}{27}.$$

Figures 1 and 2 indicate that there is a critical value of the Stokes number ($St_0 > 0$), at which J* becomes zero. When the Stokes number is less than St_0 the particles do not reach the body surface. In this case $J^* \equiv 0$ and there is no electrification of the body.

As an example we consider the electrification of a spherical body of diameter 2R = 10 m in an aerosol flow of ice particles with diameter $a = 10^{-4}$ m, concentration $\eta^0 = 10^8$ m^{-3}, and flow velocity $u^0 = 100$ m/sec. For pure ice $\varepsilon_p = 72$, $\sigma_p = 4 \cdot 10^{-7}$ $\Omega^{-1} \cdot$ m^{-1}, e = $1.6 \cdot 10^{-19}$C, $n_0 \simeq 10^{19}$–10^{20} m^{-3}, $E_p^Y = 3 \cdot 10^9$ N/m^2, $\nu_p = 0.3$. In this case the inequality $\tau \lesssim 10^{-6}$ sec $\ll \tau_e = 1.6 \cdot 10^{-3}$ sec is fulfilled and the theory expounded in Paragraph 4 is applicable. For these numerical values of the parameters we have St = 2, Re = 10^3, J* = 10^{-1}, $\Delta e_p^0 = -5 \cdot 10^{-16}$ C, $J/S_M = 5 \cdot 10^{-7}$ A/m^2. Such current densities are actually observed when bodies move in clouds and precipitation [2].

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APPLICATION OF THE MULTIPLE-SCALE METHOD IN THE PROBLEM OF WAVES ON THE SURFACE OF A LIQUID

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Sretenskii [1] has used the method of integral transforms to solve the problem of waves on the surface of a viscous incompressible liquid of inifinite depth. In the low-viscosity case Potetyunko and Strubshchik [2] have constructed asymptotic expansions that are valid in finite time intervals.

In this article we consider the planar Cauchy–Poisson problem for the linearized Navier–Stokes equations in application to the motion of an incompressible low-viscosity liquid under the action of an initial elevation of the free surface:

$$\begin{gathered} \partial \mathbf{v}/\partial t = -\nabla p + \varepsilon^2 \Delta \mathbf{v}, \ \operatorname{div} \mathbf{v} = 0, \\ p = p_r + \lambda z, \ \mathbf{v} = 0, \ \zeta = \zeta_*(x)\,(t = 0), \ -p + \lambda\zeta + 2\varepsilon^2 \partial v_z/\partial z = 0 \ (z = 0), \\ \partial\zeta/\partial t = v_z, \ \partial v_x/\partial z + \partial v_z/\partial x = 0 \ (z = 0), \\ (\mathbf{v}, \partial \mathbf{v}/\partial x, p, \partial p/\partial x, \zeta_*) \to 0, |x| \to \infty, \\ \mathbf{v} = 0 \qquad (z = -H). \end{gathered} \tag{1}$$

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All quantities in (1) are dimensionless. Here $\varepsilon^2 = 1/\mathrm{Re}$ is a small parameter; Re, Reynolds number; p_r, hydrodynamic pressure; $\zeta(x, t)$ describes the elevation profile of the free boundary; $\lambda = gT^2\alpha^{-1}$; g, acceleration of gravity; and α, T, units of length and time. The coordinate origin is placed on the undisturbed surface. The z axis is directed vertically upward. The liquid is set in motion by the initial elevation of the free boundary $\zeta_*(x)$.

We construct asymptotic expansions of the solution of problem (1) as $\varepsilon \to 0$ in the form

$$\mathbf{v} \sim \sum_{k=0}^{N} \varepsilon^k (\mathbf{v}_k + \mathbf{w}_k + \mathbf{h}_k),\ \zeta \sim \sum_{k=0}^{N} \varepsilon^k \zeta_k. \quad (2)$$

An analogous series is constructed for the function p with coefficients p_k, r_k, q_k. In the case of vanishing viscosity, boundary layers are formed near the boundaries of the domain. We denote by D_S and D_Γ the domains of the boundary layers near the solid boundary S and the free surface Γ. Then $\mathbf{w}_k$ and r_k are functions of the nature of solutions of the boundary-layer problem in D_S, while $\mathbf{h}_k$ and q_k are the same in D_Γ.

The functions $\mathbf{v}_k$ and p_k, which characterize the flow everywhere outside D_S and D_Γ, are found by the first iteration process of [3] and are expressed in terms of the scalar function $\varphi_k(x, z, t)$ according to the formulas $\mathbf{v}_k = \operatorname{grad} \varphi_k$, $p_k = -\partial\varphi_k/\partial t$, where φ_k satisfies the Laplace equation $\Delta\varphi_k = 0$. We introduce the Fourier transform with respect to the coordinate x and the Laplace transform with respect to the time t:

$$\Phi f(\xi, z, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} f(x, z, t)\, dx,\ Lf = \sigma \int_0^{\infty} f e^{-\sigma t} dt,$$

and we specify two time scales t_1 and τ [4]:

$$t_1 = t + \sum_{k=0}^{N} \varepsilon^k \beta_k(t, \xi),\ \tau = \sum_{k=1}^{N} \varepsilon^k \omega_k(t, \xi). \quad (3)$$

The principal terms of the asymptotic representation (2) $\mathbf{v}_0$, p_0, ζ_0 are determined from the solution of the corresponding ideal-fluid flow problem [3], and the coefficient ζ_0 in the expansion for the elevation of the free boundary is obtained in the form

$$\Phi\zeta_0 = \zeta^*(\xi, \tau) \cos \gamma t_1,$$

where $\gamma = (\lambda\xi \tanh \xi H)^{1/2}$ and ζ^* are expressed in terms of the initial data and τ.

The functions $\mathbf{w}_k = (w_{xk}, w_{zk})$, which occur in the domain D_S, compensate the discrepancies in the fulfillment of the no-slip conditions in (1) and are determined by means of the second iteration process in [3]. For this purpose we introduce the dilation transformation $z = -H + \varepsilon s$ and require that $\mathbf{w}_k$ and their derivatives decrease as $s \to \infty$. Then $w_{z0} = 0$, w_{x0} are determined from the heat-conduction equation with constant coefficients, and w_{z1} is obtained in the form

$$L\Phi w_{z1} = -\lambda \sqrt{\sigma}\, \xi^2 (\sigma^2 + \gamma^2)^{-1} e^{-s\sqrt{\sigma}} \zeta^*(\xi, \tau).$$

The functions ζ_1, β_1, ω_1 in the expansions (2) and (3) are determined by applying the first iteration process to the conditions on the free boundary (z = 0) in Eq. (1). As a result, for φ_1 and ζ_1 we derive the system

$$\begin{gathered} \partial\zeta_1/\partial t_1 + \Pi\zeta_0 = v_{z1},\ \partial\varphi_1/\partial t_1 + \Pi\varphi_0 + \lambda\zeta_1 = 0\ (z = 0), \\ \zeta_1 = \varphi_1 = 0\ (t_1 = 0), \end{gathered} \quad (4)$$

where the operator $\Pi = \frac{\partial\beta_1}{\partial t}\frac{\partial}{\partial t_1} + \frac{\partial\omega_1}{\partial t}\frac{\partial}{\partial\tau}$.

Separating the variables t_1 and τ in (4), we obtain the function ζ^* in the form

$$\zeta^*(\xi, \tau) = \Phi\zeta_* e^{\tau}. \quad (5)$$

Now, eliminating φ_1 from the system (4), we obtain the equation for ζ_1

$$\frac{\partial^2\Phi\zeta_1}{\partial t_1^2} + \gamma^2\Phi\zeta_1 = \left[\frac{\lambda\xi^2\sqrt{2}}{\sqrt{\gamma}\,\mathrm{ch}^2(\xi H)} M(\gamma t_1) + 2\frac{\partial\beta_1}{\partial t}\cos\gamma t_1 + 2\frac{\partial\omega_1}{\partial t}\gamma^{-1}\sin\gamma t_1\right]\zeta^*, \quad (6)$$

where

$$M(x) = \cos x \cdot c_1(\sqrt{x}) + \sin x \cdot s_1(\sqrt{x}); \; c_1(x) = \sqrt{\frac{2}{\pi}} \int_0^x \cos \xi^2 d\xi; \; s_1(x) = \sqrt{\frac{2}{\pi}} \int_0^x \sin \xi^2 d\xi.$$

From (6) we deduce ζ_1. According to the multiple-scale method [4], the unknown functions ω_1 and β_1 are determined from the condition that the coefficients of ε_1 in the asymptotic expansions (2) are bounded as $t \to \infty$; as a result, we have

$$\beta_1 = \beta t, \; \omega_1 = \gamma\beta t, \; \beta = -\lambda\xi^2/[2\sqrt{2}\gamma^2\sqrt{\gamma}\,\mathrm{ch}^2(\xi H)].$$

For the first two terms of the asymptotic expansion of the elevation of the free surface ζ we have

$$\zeta \sim \zeta_0 + \varepsilon\zeta_1, \; \Phi\zeta_0 = \cos\gamma(1+\varepsilon\beta)\,t\cdot\Phi\zeta_* e^{-\varepsilon\gamma\beta t},$$

$$\Phi\zeta_1 = \frac{\lambda\xi^2}{2\sqrt{2}\,\gamma^2\sqrt{\gamma}\,\mathrm{ch}^2(\xi H)}\left[\gamma t(\cos\gamma t - \sin\gamma t) - \sqrt{\frac{2\gamma t}{\pi}} + M(\gamma t) + 2\gamma t\left(c_1(\sqrt{\gamma t})\sin\gamma t - s_1(\sqrt{\gamma t})\cos\gamma t\right)\right]\Phi\zeta_* e^{-\varepsilon\gamma\beta t}. \quad (7)$$

We consider the case of a liquid of infinite depth. Now the last condition in the system (1) is satisfied for $z = -\infty$, and the coefficients w_k and r_k are absent in the expansions (2). The expression for the elevation ζ_0 of the free boundary of an ideal fluid is obtained in the form $\Phi\zeta_0 = \cos\varphi t_1 \cdot \zeta^*(\xi, \tau)$, where $\varphi = (\lambda|\xi|)^{1/2}$ and ζ^* is determined according to (5).

As $\varepsilon \to 0$ a boundary layer is formed only near the free boundary Γ. The functions $\mathbf{h}_k = (h_{xk}, h_{zk})$ compensate the discrepancy in the fulfillment of the dynamic condition for the tangential stress on Γ and are determined by the second iteration process, where $\mathbf{h}_0 = h_{z1} = q_k = 0$ $(k \ge 0)$: $L\Phi h_{z2} = -2\lambda\xi^3(\sigma^2+\varphi^2)^{-1}\exp(-s\sqrt{\sigma})\zeta^*$, $s = z/\varepsilon$. The coefficients β_2 and ω_2 in the expansions (3) are calculated concurrently with the determination of the functions $\mathbf{v}_2$, ζ_2, p_2. We note that in the given situation $\mathbf{v}_1 = p_1 = \zeta_1 = \omega_1 = \beta_1 = 0$, and ζ_2 is determined from the equation

$$\Phi\zeta_2 = \left[(\varphi\beta_2 + 2\xi^2\varphi^{-1})\sin\varphi t_1 - (\omega_2 + 2\xi^2 t_1)\cos\varphi t_1\right]e^{\tau}\Phi\zeta_*.$$

From the condition of boundedness of the coefficients of ε^2 in the expansions (2) as $t \to \infty$ we deduce expressions for β_2 and ω_2: $\omega_2 = -2\xi^2 t$, $\beta_2 = 0$.

For the asymptotic expansion of the elevation of the free boundary, up to terms of order ε^3, we obtain

$$\zeta = \int_{-\infty}^{\infty} e^{-2\varepsilon^2\xi^2 t}\,\Phi\zeta_*\left(\cos\varphi t + 2\varepsilon^2\xi^2\varphi^{-1}\sin\varphi t\right)e^{i\xi x}d\xi + O(\varepsilon^3). \quad (8)$$

The constructed asymptotic expansions (2) and expressions (7) and (8) describe the attenuation of the waves generated by the initial disturbance of the free surface at times of the order $O[(\mathrm{Re})^{1/2}]$ and $O(\mathrm{Re})$, respectively. We note that the coefficient of ε^0 in the asymptotic formulas (8) coincides with the well-known integral of Sretenskii [1], and upon expansion of the exponential function $\exp(-2\varepsilon^2\xi^2 t)$ into a power series in ε the first two terms of the asymptotic representation of ζ in [2] are obtained.

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